

ON EQUIVARIANT MAPS RELATED TO THE SPACE OF PAIRS OF EXCEPTIONAL JORDAN ALGEBRAS

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ABSTRACT. Let \mathcal{J} be the exceptional Jordan algebra and $V = \mathcal{J} \oplus \mathcal{J}$. We construct an equivariant map from V to $\text{Hom}_k(\mathcal{J} \otimes \mathcal{J}, \mathcal{J})$ defined by homogeneous polynomials of degree 8 such that if $x \in V$ is a generic point, then the image of x is the structure constant of the isotope of \mathcal{J} corresponding to x . We also give an alternative way to define the isotope corresponding to a generic point of \mathcal{J} by an equivariant map from \mathcal{J} to the space of trilinear forms.

1. INTRODUCTION

Let k be a field of characteristic not equal to 2, 3, k^{sep} the separable closure of k and \bar{k} the algebraic closure of k . Let $\tilde{\mathcal{O}}$ be the split octonion over k . It is the normed algebra over k obtained by the Cayley–Dickson process (see [2, pp.101–110]). If A is the algebra of 2×2 matrices, $\tilde{\mathcal{O}}$ is $A(+)$ with the notation of [2]. An *octonion* is, by definition, a normed algebra which is a k -form of $\tilde{\mathcal{O}}$. Let \mathcal{O} be an octonion. We use the notation $\|x\|$ for the norm of $x \in \mathcal{O}$. If $a \in k$, $\|ax\| = a^2\|x\|$. Also if $x, y \in \mathcal{O}$, then $\|xy\| = \|x\|\|y\|$. For $x, y \in \mathcal{O}$, let

$$Q(x, y) = \frac{1}{2}(\|x + y\| - \|x\| - \|y\|).$$

This is a non-degenerate symmetric bilinear form such that $Q(x, x) = \|x\|$. Let $W \subset \mathcal{O}$ be the orthogonal complement of $k \cdot 1$ with respect to Q . If $x = x_1 + x_2$ where $x_1 \in k \cdot 1$, $x_2 \in W$, then we define $\bar{x} = x_1 - x_2$ and call it the *conjugate* of x . Note that $\|x\| = x\bar{x}$. For $x \in \mathcal{O}$, we define the trace $\text{tr}(x)$ by $\text{tr}(x) = x + \bar{x}$. It is easy to verify that

$$\text{tr}(xy) = \text{tr}(yx), \quad 2Q(x, y) = \text{tr}(x\bar{y}).$$

Let $\text{GL}(n)$ be the group of $n \times n$ invertible matrices over k . If V is a finite dimensional vector space, then we denote the group of invertible linear maps from V to V by $\text{GL}(V)$. Let \mathcal{J} be the exceptional Jordan algebra over k . Elements of \mathcal{J} are of the form:

$$X = \begin{pmatrix} s_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & s_2 & x_1 \\ x_2 & \bar{x}_1 & s_3 \end{pmatrix}, \quad s_i \in k, \quad x_i \in \mathcal{O} \quad (i = 1, 2, 3),$$

The multiplication of \mathcal{J} is defined as follows:

$$X \circ Y = \frac{1}{2}(XY + YX),$$

where the multiplication used on the right-hand side is the multiplication of matrices.

The algebraic groups E_6 and GE_6 are given by

$$E_6 = \{L \in \text{GL}(\mathcal{J}) \mid \forall X \in \mathcal{J}, \det(LX) = \det(X)\},$$

$$GE_6 = \{L \in \text{GL}(\mathcal{J}) \mid \forall X \in \mathcal{J}, \det(LX) = c(L) \det(X) \text{ for some } c(L) \in \text{GL}(1)\}$$

respectively. Then $c : GE_6 \rightarrow \text{GL}(1)$ is a character and there exists an exact sequence

$$(1.1) \quad 0 \rightarrow E_6 \hookrightarrow GE_6 \xrightarrow{c} \text{GL}(1) \rightarrow 0.$$

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It is known that E_6 is a smooth connected quasi-simple simply-connected algebraic group of type E_6 (see [5, p.181, Theorem 7.3.2]). The terminology “quasi-simple” means that its inner automorphism group is simple (see [4, p.136]). The smoothness of the group follows from the fact that the dimension of E_6 as a variety and the dimension of the Lie algebra of E_6 coincide (see the proof of [5, p.181, Theorem 7.3.2]).

Let $H_1 = E_6$, $G_1 = GE_6$, $H = H_1 \times \mathrm{GL}(2)$ and $G = G_1 \times \mathrm{GL}(2)$. Let $V = \mathcal{J} \otimes \mathrm{Aff}^2$. We regard elements of V as the set of $x = x_1v_1 + x_2v_2$ where $x_1, x_2 \in \mathcal{J}$ and two variables v_1, v_2 . The action of $g = (g_1, g_2) \in G$ where $g_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on V is given by

$$(1.2) \quad g(x_1v_1 + x_2v_2) = g_1(x_1)(av_1 + cv_2) + g_1(x_2)(bv_1 + dv_2).$$

For $x = x_1v_1 + x_2v_2 \in V$, we put $F_x(v) = F_x(v_1, v_2) = \det(x)$. Then $F_x(v)$ is a binary cubic form. Let $\Delta(x)$ be the discriminant of $F_x(v)$ as a polynomial of v . Let

$$(1.3) \quad w = w_1v_1 + w_2v_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} v_1 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} v_2 \in \mathcal{J} \otimes \mathrm{Aff}^2.$$

Then it is easy to see that $F_w(v) = v_1v_2(v_1 - v_2)$ and $\Delta(w) = 1$. The pair (G, V) is an irreducible regular *prehomogeneous vector space* (see [3, Proposition 3.2] and [6]) and the polynomial $\Delta(x)$ is what we call a *relative invariant polynomial*. We define $V^{\mathrm{ss}} = \{x \in V \mid \Delta(x) \neq 0\}$. Points in V^{ss} are called *semi-stable points*. As we pointed out as above, $w \in V_k^{\mathrm{ss}}$. The polynomial $\Delta(x)$ is of degree 12. If we put $\chi(g) = c(g_1)^4 \det(g_2)^6$ for $g = (g_1, g_2) \in G$, then $\Delta(gx) = \chi(g)\Delta(x)$.

A Jordan algebra \mathcal{M} is called an isotope of \mathcal{J} if $\mathcal{M} \otimes k^{\mathrm{sep}} \cong \mathcal{J} \otimes k^{\mathrm{sep}}$ and the “determinant” of \mathcal{M} is a constant multiple of that of \mathcal{J} . For details, the reader should see [5, pp.154–158]. If $\mathcal{M}_1, \mathcal{M}_2$ are isotopes of \mathcal{J} and $\mathfrak{n}_1 \subset \mathcal{M}_1, \mathfrak{n}_2 \subset \mathcal{M}_2$ are cubic étale subalgebras, then the pairs $(\mathcal{M}_1, \mathfrak{n}_1), (\mathcal{M}_2, \mathfrak{n}_2)$ are defined to be equivalent if there exists a k -isomorphism $\mathcal{M}_1 \rightarrow \mathcal{M}_2$ which induces an isomorphism from \mathfrak{n}_1 to \mathfrak{n}_2 . Let $\mathrm{JIC}(k)$ be the set of equivalence classes of pairs $(\mathcal{M}, \mathfrak{n})$ as above.

In [3], to each point in V_k^{ss} , a pair $(\mathcal{M}, \mathfrak{n}) \in \mathrm{JIC}(k)$ was associated. It is proved in [3, Theorem 5.8] that there is a bijective correspondence between the set $G_k \backslash V_k^{\mathrm{ss}}$ of rational orbits and $\mathrm{JIC}(k)$. Moreover, an equivariant map $m : V \rightarrow \mathcal{J}$ was defined and the Jordan algebra corresponding to $x \in V_k^{\mathrm{ss}}$ was explicitly described using the point $m(x)$ (see [3, Section 5]).

Let $\mathfrak{t} \subset \mathcal{J}$ be the subalgebra of diagonal matrices, which is isomorphic to k^3 . Let $\mathrm{Aut}(\mathcal{J})$ be the algebraic group of automorphisms of the Jordan algebra \mathcal{J} . We define $\mathrm{Aut}(\mathcal{J}, \mathfrak{t})$ to be the subgroup of $\mathrm{Aut}(\mathcal{J})$ consisting of automorphisms L such that $L(\mathfrak{t}) = \mathfrak{t}$. It is proved in ([3, Theorem 3.1, Lemma 4.2]) that $G_w \cong \mathrm{GL}(1) \times \mathrm{Aut}(\mathcal{J}, \mathfrak{t})$. The first Galois cohomology set $H^1(k, \mathrm{GL}(1) \times \mathrm{Aut}(\mathcal{J}, \mathfrak{t}))$ can be identified with $H^1(k, \mathrm{Aut}(\mathcal{J}, \mathfrak{t}))$ and there is a natural map $H^1(k, \mathrm{Aut}(\mathcal{J}, \mathfrak{t})) \rightarrow H^1(k, \mathrm{Aut}(\mathcal{J}))$. One can show by standard argument that elements of $H^1(k, \mathrm{Aut}(\mathcal{J}))$ correspond bijectively with k -forms of \mathcal{J} .

Suppose that $x = g_x w \in V_k^{\mathrm{ss}}$ where $g_x \in G_{k^{\mathrm{sep}}}$. Then $h_x : \mathrm{Gal}(k^{\mathrm{sep}}/k) \ni \sigma \mapsto g_x^{-1} g_x^\sigma \in G_{w, k^{\mathrm{sep}}}$ is a 1-cocycle, which defines an element, say c_x of $H^1(k, G_w)$. This element c_x does not depend on the choice of g_x . Since there is a natural map $H^1(k, G_w) \cong H^1(k, \mathrm{Aut}(\mathcal{J}, \mathfrak{t})) \rightarrow H^1(k, \mathrm{Aut}(\mathcal{J}))$ (\cong means bijection), c_x determines a k -form of \mathcal{J} . If x corresponds to a pair $(\mathcal{M}, \mathfrak{n})$, then \mathcal{M} is the k -form of \mathcal{J} which is determined by c_x . The underlying vector space of \mathcal{M} is \mathcal{J} . To define a Jordan algebra structure on \mathcal{J} , it is enough to define the product structure, which is given by an element of $\mathrm{Hom}_k(\mathcal{J} \otimes \mathcal{J}, \mathcal{J})$ (we call this element the “structure constant”).

If we choose bases of V, \mathcal{J} as k -vector spaces, the map $m : V \rightarrow \mathcal{J}$ is given by homogeneous polynomials of degree 4 on V . The structure constant which is associated to elements of \mathcal{J} is, with the denominator multiplied, given by homogeneous polynomials of degree 11 on \mathcal{J} . So the structure constant of \mathcal{M} is given by homogeneous polynomials of degree 44 on V .

The first purpose of this paper is to prove the following theorem (see Section 2).

Theorem 1.4. *There is an equivariant map $S : V \ni x \mapsto S_x \in \mathrm{Hom}_k(\mathcal{J} \otimes \mathcal{J}, \mathcal{J})$ defined by homogeneous polynomials of degree 8 on V such that if $x \in V_k^{\mathrm{ss}}$ corresponds to the pair $(\mathcal{M}, \mathfrak{n}) \in \mathrm{JIC}(k)$, then $\Delta(x)^{-1} S_x$ is the structure constant of the Jordan algebra \mathcal{M} .*

Let $a \in \mathcal{J}$ and $\det(a) \neq 0$. In [5, p. 155], for $X, Y \in \mathcal{J}$, the product structure $X \circ_a Y$ defining the isotope \mathcal{J}_a and the corresponding symmetric bilinear form $Q_a(X, Y)$ are given (see (3.5)). Then $\mathcal{J}^3 \ni (X, Y, Z) \mapsto T_a(X, Y, Z) = Q_a(X \circ_a Y, Z)$ is a trilinear form on \mathcal{J} . To construct T_a first equivariantly to provide an alternative way to define the product structure on \mathcal{J}_a is another purpose of this paper.

We prove the following theorem in Section 3.

Theorem 1.5. *There is an equivariant map $T : \mathcal{J} \ni a \mapsto T_a \in \text{Hom}_k(\mathcal{J} \otimes \mathcal{J} \otimes \mathcal{J}, k)$ defined by homogeneous polynomials of degree 6 such that if for $X, Y \in \mathcal{J}$, $X \circ_a Y \in \mathcal{J}$ is the element such $Q_a(X \circ_a Y, Z) = \det(a)^{-1} T_a(X, Y, Z)$ for all $Z \in \mathcal{J}$, then this product structure coincides with that of the isotope \mathcal{J}_a .*

2. EQUIVARIANT MAP I

We prove Theorem 1.4 in this section. We first define basic notions and then define the desired equivariant map.

We denote the 3×3 diagonal matrix with diagonal entries $\alpha_1, \alpha_2, \alpha_3 \in k^\times$ by $\text{diag}(\alpha_1, \alpha_2, \alpha_3)$. $\mathcal{J}^{n \otimes}$ is the tensor product of n copies of \mathcal{J} . We define a symmetric trilinear form D on \mathcal{J} by

$$6D(X, Y, Z) = \det(X + Y + Z) - \det(X + Y) - \det(Y + Z) - \det(Z + X) \\ + \det(X) + \det(Y) + \det(Z)$$

for $X, Y, Z \in \mathcal{J}$. Let $\text{Tr}(X)$ be the sum of diagonal entries of $X \in \mathcal{J}$. For $X, Y \in \mathcal{J}$, we define a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on \mathcal{J} by

$$\langle X, Y \rangle = \text{Tr}(X \circ Y).$$

One can verify by direct computation that the symmetric bilinear form $\langle \cdot, \cdot \rangle$ satisfies the following equation:

$$(2.1) \quad \langle X \circ Y, Z \rangle = \langle X, Y \circ Z \rangle, \quad \forall X, Y, Z \in \mathcal{J}.$$

For $X, Y \in \mathcal{J}$, the cross product $X \times Y$ is, by definition, the element satisfying the following equation:

$$\langle X \times Y, Z \rangle = 3D(X, Y, Z), \quad \forall Z \in \mathcal{J}.$$

Let $e = I_3$. Then the following equations are satisfied (see [5, p.122, Lemma 5.2.1]).

$$(2.2) \quad \forall X \in \mathcal{J}, \quad X \circ (X \times X) = \det(X)e, \quad e \times e = e.$$

For any $g \in \text{GL}(\mathcal{J})$, we define $\tilde{g} \in \text{GL}(\mathcal{J})$ by

$$\langle g(X), \tilde{g}(Y) \rangle = \langle X, Y \rangle, \quad \forall X, Y \in \mathcal{J}.$$

The following lemma is proved in [5, p.180, Proposition 7.3.1].

Lemma 2.3. *The map $g \mapsto \tilde{g}$ is an automorphism of H_1 with order 2 and for all $X, Y \in \mathcal{J}$,*

$$g(X \times Y) = \tilde{g}(X) \times \tilde{g}(Y), \quad \tilde{g}(X \times Y) = g(X) \times g(Y).$$

Corollary 2.4. *If $g \in G_1$ and $X, Y, Z, W \in \mathcal{J}$, then*

$$g((X \times Y) \times (Z \times W)) = c(g)^{-1} (g(X) \times g(Y)) \times (g(Z) \times g(W)).$$

Proof. There exists $t \in \bar{k}$ such that $t^3 = c(g_1)$. Then $t^{-1}g \in H_{1\bar{k}}$. So

$$g((X \times Y) \times (Z \times W)) = t(t^{-1}g)((X \times Y) \times (Z \times W)) \\ = t(t^{-1}g(X) \times t^{-1}g(Y)) \times (t^{-1}g(Z) \times t^{-1}g(W)) \\ = c(g)^{-1} (g(X) \times g(Y)) \times (g(Z) \times g(W)).$$

□

We now define an equivariant map

$$S : V \ni x \mapsto S_x \in \text{Hom}_k(\mathcal{J} \otimes \mathcal{J}, \mathcal{J})$$

such that $S_w(X, Y) = X \circ Y$ for all $X, Y \in \mathcal{J}$.

Let

$$(2.5) \quad X = \begin{pmatrix} s_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & s_2 & x_1 \\ x_2 & \bar{x}_1 & s_3 \end{pmatrix}, \quad Y = \begin{pmatrix} t_1 & y_3 & \bar{y}_2 \\ \bar{y}_3 & t_2 & y_1 \\ y_2 & \bar{y}_1 & t_3 \end{pmatrix}$$

where $s_i, t_i \in k, x_i, y_i \in \mathcal{O}$ for $i = 1, 2, 3$. By computation, $X \circ Y$ is the following matrix:

$$(2.6) \quad \frac{1}{2} \begin{pmatrix} 2s_1t_1 + \text{tr}(x_3\bar{y}_3 + x_2\bar{y}_2) & s_1y_3 + t_2x_3 + \bar{x}_2\bar{y}_1 & s_1\bar{y}_2 + x_3y_1 + t_3\bar{x}_2 \\ & +t_1x_3 + s_2y_3 + \bar{y}_2\bar{x}_1 & +t_1\bar{x}_2 + y_3x_1 + s_3\bar{y}_2 \\ s_1\bar{y}_3 + t_2\bar{x}_3 + y_1x_2 & 2s_2t_2 + \text{tr}(x_3\bar{y}_3 + x_1\bar{y}_1) & \bar{x}_3\bar{y}_2 + s_2y_1 + t_3x_1 \\ +t_1\bar{x}_3 + s_2\bar{y}_3 + x_1y_2 & & +\bar{y}_3\bar{x}_2 + t_2x_1 + s_3y_1 \\ s_1y_2 + \bar{y}_1\bar{x}_3 + t_3x_2 & y_2x_3 + s_2\bar{y}_1 + t_3\bar{x}_1 & 2s_3t_3 + \text{tr}(x_2\bar{y}_2 + x_1\bar{y}_1) \\ +t_1x_2 + \bar{x}_1\bar{y}_3 + s_3y_2 & +x_2y_3 + t_2\bar{x}_1 + s_3\bar{y}_1 & \end{pmatrix}.$$

Note that $\text{tr}(\bar{x}_2y_2) = \text{tr}(y_2\bar{x}_2) = \text{tr}(x_2\bar{y}_2)$, etc. In particular, if Y is diagonal, then

$$(2.7) \quad X \circ Y = \frac{1}{2} \begin{pmatrix} 2s_1t_1 & (t_1 + t_2)x_3 & (t_1 + t_3)\bar{x}_2 \\ (t_1 + t_2)\bar{x}_3 & 2s_2t_2 & (t_2 + t_3)x_1 \\ (t_1 + t_3)x_2 & (t_2 + t_3)\bar{x}_1 & 2s_3t_3 \end{pmatrix}.$$

It is known (see [5, p.122]) that

$$(2.8) \quad \begin{aligned} X \times Y &= X \circ Y - \frac{1}{2}\langle X, e \rangle Y - \frac{1}{2}\langle Y, e \rangle X - \frac{1}{2}\langle X, Y \rangle e + \frac{1}{2}\langle X, e \rangle \langle Y, e \rangle e \\ &= X \circ Y - \frac{1}{2}\text{Tr}(X)Y - \frac{1}{2}\text{Tr}(Y)X - \frac{1}{2}\text{Tr}(X \circ Y)e + \frac{1}{2}\text{Tr}(X)\text{Tr}(Y)e. \end{aligned}$$

For $a, b \in \mathcal{J}$ and $i_1, \dots, i_8 = 1, 2$, we define

$$(2.9) \quad [[i_1, i_2 | i_3, i_4 | i_5, i_6 | i_7, i_8]]_{(a,b)} = v_1 \otimes \cdots \otimes v_8$$

where $v_j = a$ (resp. $v_j = b$) if $i_j = 1$ (resp. $i_j = 2$). For example,

$$[[1, 2 | 2, 1 | 2, 1 | 1, 2]]_{(a,b)} = a \otimes b \otimes b \otimes a \otimes b \otimes a \otimes a \otimes b.$$

We only consider $[[i_1, i_2 | i_3, i_4 | i_5, i_6 | i_7, i_8]]_{(a,b)}$ such that $\{i_{2j-1}, i_{2j}\} = \{1, 2\}$ ($j = 1, \dots, 4$).

Let

$$t_4(a, b) = \overbrace{(a \otimes b - b \otimes a) \otimes \cdots \otimes (a \otimes b - b \otimes a)}^4 \in \mathcal{J}^{8 \otimes}.$$

Note that $t_4(a, b)$ depends only on $a \wedge b \in \wedge^2 \mathcal{J}$. By expanding all the terms,

$$\begin{aligned} t_4(a, b) &= [[1, 2 | 1, 2 | 1, 2 | 1, 2]]_{(a,b)} - [[1, 2 | 1, 2 | 1, 2 | 2, 1]]_{(a,b)} \\ &\quad - [[1, 2 | 1, 2 | 2, 1 | 1, 2]]_{(a,b)} + [[1, 2 | 1, 2 | 2, 1 | 2, 1]]_{(a,b)} \\ &\quad - [[1, 2 | 2, 1 | 1, 2 | 1, 2]]_{(a,b)} + [[1, 2 | 2, 1 | 1, 2 | 2, 1]]_{(a,b)} \\ &\quad + [[1, 2 | 2, 1 | 2, 1 | 1, 2]]_{(a,b)} - [[1, 2 | 2, 1 | 2, 1 | 2, 1]]_{(a,b)} \\ &\quad - [[2, 1 | 1, 2 | 1, 2 | 1, 2]]_{(a,b)} + [[2, 1 | 1, 2 | 1, 2 | 2, 1]]_{(a,b)} \\ &\quad + [[2, 1 | 1, 2 | 2, 1 | 1, 2]]_{(a,b)} - [[2, 1 | 1, 2 | 2, 1 | 2, 1]]_{(a,b)} \\ &\quad + [[2, 1 | 2, 1 | 1, 2 | 1, 2]]_{(a,b)} - [[2, 1 | 2, 1 | 1, 2 | 2, 1]]_{(a,b)} \\ &\quad - [[2, 1 | 2, 1 | 2, 1 | 1, 2]]_{(a,b)} + [[2, 1 | 2, 1 | 2, 1 | 2, 1]]_{(a,b)}. \end{aligned}$$

For $v_1, \dots, v_8, X, Y \in \mathcal{J}$, we put

$$\begin{aligned} \Psi_1(v_1 \otimes \cdots \otimes v_8)(X, Y) &= D(v_2, v_5, v_7)D(v_4, v_6, v_8)(v_1 \times v_3) \times (X \times Y), \\ \Psi_2(v_1 \otimes \cdots \otimes v_8)(X, Y) &= D(v_2, v_5, v_7)D(v_4, v_6, v_8)(D(v_1, v_3, X)Y + D(v_1, v_3, Y)X). \end{aligned}$$

Then Ψ_i induces a k -linear map $\mathcal{J}^{8\otimes} \rightarrow \text{Hom}_k(\mathcal{J} \otimes \mathcal{J}, \mathcal{J})$ for $i = 1, 2$. We define

$$(2.10) \quad \Phi_{i,(a,b)}(X, Y) = \Psi_i(t_4(a, b))(X, Y)$$

for $i = 1, 2$. Then

$$\begin{array}{ccccc} \Phi_i : & V & \rightarrow & (\wedge \mathcal{J}^2)^{4\otimes} & \rightarrow & \text{Hom}_k(\mathcal{J} \otimes \mathcal{J}, \mathcal{J}) \\ & \downarrow & & \downarrow & & \downarrow \\ & (a, b) & \mapsto & t_4(a, b) & \mapsto & \Phi_{i,(a,b)} \end{array}$$

is a k -linear map defined by degree 8 polynomials of (a, b) .

Lemma 2.11. $\Phi_{i,gx}(g_1X, g_1Y) = c(g_1)^3 \det(g_2)^4 g_1(\Phi_{i,x}(X, Y))$ for $i = 1, 2$ and all $g = (g_1, g_2) \in G$, $X, Y \in \mathcal{J}$.

Proof. If $x = (x_1, x_2)$, then $t_4(x_1, x_2)$ depends only on $x_1 \wedge x_2$. If $g_2 \in \text{GL}(2)$ and $y = (y_1, y_2) = gx$, then $y_1 \wedge y_2 = (\det(g_2))x_1 \wedge x_2$. Since $x_1 \otimes x_2 - x_2 \otimes x_1$ can be identified with $x_1 \wedge x_2$ and it appears in the definition of $t_4(x_1, x_2)$ four times, $t_4(y_1, y_2) = (\det g_2)^4 t_4(x_1, x_2)$. So we may assume that $g_2 = 1$ and only consider the action of G_1 .

Suppose that $v_1 \otimes \cdots \otimes v_8$ is a term which appears in the expansion of $t_4(x_1, x_2)$. If $g_1 \in G_1$ and x is replaced by $g_1x = (g_1x_1, g_1x_2)$, then terms which appear in the expansion of $t_4(g_1x_1, g_1x_2)$ are in the form $g_1v_1 \otimes \cdots \otimes g_1v_8$. By Corollary 2.4,

$$\begin{aligned} & D(g_1v_2, g_1v_5, g_1v_7)D(g_1v_4, g_1v_6, g_1v_8)(g_1v_1 \times g_1v_3) \times (g_1X \times g_1Y), \\ & = c(g_1)^3 D(v_2, v_5, v_7)D(v_4, v_6, v_8)g_1((v_1 \times v_3) \times (X \times Y)). \end{aligned}$$

Also

$$\begin{aligned} & D(g_1v_2, g_1v_5, g_1v_7)D(g_1v_4, g_1v_6, g_1v_8)(D(g_1v_1, g_1v_3, g_1X)g_1Y + D(g_1v_1, g_1v_3, g_1Y)g_1X) \\ & = c(g_1)^3 D(v_2, v_5, v_7)D(v_4, v_6, v_8)g_1(D(v_1, v_3, X)Y + D(v_1, v_3, Y)X). \end{aligned}$$

Therefore, $\Phi_{i,g_1x}(g_1X, g_1Y) = c(g_1)^3 g_1(\Phi_{i,x}(X, Y))$ for $i = 1, 2$. □

We first evaluate $\Phi_{1,w}$ (see (1.3)).

Proposition 2.12. For all $X, Y \in \mathcal{J}$, $-18\Phi_{1,w}(X, Y) = X \circ Y - \frac{1}{2}\text{Tr}(Y)X - \frac{1}{2}\text{Tr}(X)Y$.

Proof. Suppose that $(a, b) = w = (w_1, w_2)$ in the following. If we expand $t_4(a, b)$, then the coefficient of $[[i_1, i_2|i_3, i_4|i_5, i_6|i_7, i_8]]_{(a,b)}$ is 1 (resp. -1) if the number of $j = 1, \dots, 4$ such that $(i_{2j-1}, i_{2j}) = (2, 1)$ is even (resp. odd). We list the sign in $t_4(a, b)$, $D(v_2, v_5, v_7)D(v_4, v_6, v_8)$ and $v_1 \times v_3$ in the following table assuming that $[[i_1, i_2|i_3, i_4|i_5, i_6|i_7, i_8]]_{(a,b)}$ is in the form (2.9).

(1)	$D(v_2, v_5, v_7)D(v_4, v_6, v_8)$	$v_1 \times v_3$
$+[[1, 2 1, 2 1, 2 1, 2]]$	$D(b, a, a)D(b, b, b)$	$a \times a$
$-[[1, 2 1, 2 1, 2 2, 1]]$	$D(b, a, b)D(b, b, a)$	$a \times a$
$-[[1, 2 1, 2 2, 1 1, 2]]$	$D(b, b, a)D(b, a, b)$	$a \times a$
$+[[1, 2 1, 2 2, 1 2, 1]]$	$D(b, b, b)D(b, a, a)$	$a \times a$
$-[[1, 2 2, 1 1, 2 1, 2]]$	$D(b, a, a)D(a, b, b)$	$a \times b$
$+[[1, 2 2, 1 1, 2 2, 1]]$	$D(b, a, b)D(a, b, a)$	$a \times b$
$+[[1, 2 2, 1 2, 1 1, 2]]$	$D(b, b, a)D(a, a, b)$	$a \times b$
$-[[1, 2 2, 1 2, 1 2, 1]]$	$D(b, b, b)D(a, a, a)$	$a \times b$
$-[[2, 1 1, 2 1, 2 1, 2]]$	$D(a, a, a)D(b, b, b)$	$b \times a$
$+[[2, 1 1, 2 1, 2 2, 1]]$	$D(a, a, b)D(b, b, a)$	$b \times a$
$+[[2, 1 1, 2 2, 1 1, 2]]$	$D(a, b, a)D(b, a, b)$	$b \times a$
$-[[2, 1 1, 2 2, 1 2, 1]]$	$D(a, b, b)D(b, a, a)$	$b \times a$
$+[[2, 1 2, 1 1, 2 1, 2]]$	$D(a, a, a)D(a, b, b)$	$b \times b$
$-[[2, 1 2, 1 1, 2 2, 1]]$	$D(a, a, b)D(a, b, a)$	$b \times b$
$-[[2, 1 2, 1 2, 1 1, 2]]$	$D(a, b, a)D(a, a, b)$	$b \times b$
$+[[2, 1 2, 1 2, 1 2, 1]]$	$D(a, b, b)D(a, a, a)$	$b \times b$

((1) is $[[i_1, i_2 | i_3, i_4 | i_5, i_6 | i_7, i_8]]_{(a,b)}$ with its sign in $t_4(a, b)$.)

Note that if $(a, b) = w$, then

$$D(a, a, a) = D(b, b, b) = 0, \quad D(a, a, b) = \frac{1}{3}, \quad D(a, b, b) = -\frac{1}{3}.$$

So we can ignore terms with the second column including either $D(a, a, a)$ or $D(b, b, b)$. Removing these terms, we obtain the following table.

(1)	$9D(v_2, v_5, v_7)D(v_4, v_6, v_8)$	$v_1 \times v_3$
$-[[1, 2 1, 2 1, 2 2, 1]]$	1	$a \times a$
$-[[1, 2 1, 2 2, 1 1, 2]]$	1	$a \times a$
$-[[1, 2 2, 1 1, 2 1, 2]]$	-1	$a \times b$
$+[[1, 2 2, 1 1, 2 2, 1]]$	-1	$a \times b$
$+[[1, 2 2, 1 2, 1 1, 2]]$	-1	$a \times b$
$+[[2, 1 1, 2 1, 2 2, 1]]$	-1	$b \times a$
$+[[2, 1 1, 2 2, 1 1, 2]]$	-1	$b \times a$
$-[[2, 1 1, 2 2, 1 2, 1]]$	-1	$b \times a$
$-[[2, 1 2, 1 1, 2 2, 1]]$	1	$b \times b$
$-[[2, 1 2, 1 2, 1 1, 2]]$	1	$b \times b$

By the above table,

$$9\Phi_{1,w}(X, Y) = -(2a \times a + a \times b + a \times b + 2b \times b) \times (X \times Y).$$

If we put $c = a + b = \text{diag}(1, 0, -1)$, then

$$9\Phi_{1,w}(X, Y) = -(a \times a + b \times b + c \times c) \times (X \times Y).$$

By (2.7) and (2.8)

$$a \times a = \text{diag}(0, 0, -1), \quad b \times b = \text{diag}(-1, 0, 0), \quad c \times c = \text{diag}(0, -1, 0).$$

Therefore, $9\Phi_{1,w}(X, Y) = e \times (X \times Y)$. Since e is the unit element of \mathcal{J} , replacing X, Y in (2.8) by $e, X \times Y$, we obtain

$$\begin{aligned} 9\Phi_{1,w}(X, Y) &= X \times Y - \frac{1}{2}\text{Tr}(e)X \times Y - \frac{1}{2}\text{Tr}(X \times Y)e - \frac{1}{2}\text{Tr}(X \times Y)e + \frac{1}{2}\text{Tr}(e)\text{Tr}(X \times Y)e \\ &= -\frac{1}{2}X \times Y + \frac{1}{2}\text{Tr}(X \times Y)e. \end{aligned}$$

By (2.8),

$$\begin{aligned} \text{Tr}(X \times Y) &= \text{Tr}(X \circ Y) - \text{Tr}(X)\text{Tr}(Y) - \frac{3}{2}\text{Tr}(X \circ Y) + \frac{3}{2}\text{Tr}(X)\text{Tr}(Y) \\ &= -\frac{1}{2}\text{Tr}(X \circ Y) + \frac{1}{2}\text{Tr}(X)\text{Tr}(Y). \end{aligned}$$

Therefore, again by (2.8),

$$(2.13) \quad 9\Phi_{1,w}(X, Y) = -\frac{1}{2}X \circ Y + \frac{1}{4}\text{Tr}(Y)X + \frac{1}{4}\text{Tr}(X)Y.$$

Multiplying -2 , we obtain the proposition. □

Proposition 2.14. For all $X, Y \in \mathcal{J}$, $3\Phi_{2,w}(X, Y) = \text{Tr}(Y)X + \text{Tr}(X)Y$.

Proof. By very similar computations as in the case of $\Phi_{1,w}(X, Y)$, we obtain

$$\begin{aligned} (2.15) \quad 3\Phi_{2,w}(X, Y) &= -3(D(a, a, X) + D(b, b, X) + D(c, c, X))Y \\ &\quad - 3(D(a, a, Y) + D(b, b, Y) + D(c, c, Y))X \\ &= (s_3 + s_1 + s_2)Y + (t_3 + t_1 + t_2)X \\ &= \text{Tr}(Y)X + \text{Tr}(X)Y. \end{aligned}$$

□

For $(a, b) \in V_k$, we define

$$(2.16) \quad S_{(a,b)}(X, Y) = -18\Phi_{1,(a,b)}(X, Y) + \frac{3}{2}\Phi_{2,(a,b)}(X, Y).$$

The following proposition follows from Lemma 2.11 and (2.13), (2.15).

Proposition 2.17. (1) If $x \in V_k$, $g = (g_1, g_2) \in G_k$ and $X, Y \in \mathcal{J}$, then

$$S_{gx}(g_1X, g_1Y) = c(g_1)^3 \det(g_2)^4 g_1(S_x(X, Y)).$$

(2) $S_w(X, Y) = X \circ Y$.

Let $\Delta(x)$ be the relative invariant polynomial of degree 12 and $\Delta(w) = 1$ which we defined in Introduction. For $x \in V_k^{\text{ss}}$, we define a k -algebra structure \circ_x on \mathcal{J} by

$$(2.18) \quad X \circ_x Y \stackrel{\text{def}}{=} \Delta(x)^{-1} S_x(X, Y).$$

We denote this k -algebra on the underlying vector space \mathcal{J} by \mathcal{M}_x . Proposition 2.17 (2) implies that \mathcal{M}_w is isomorphic to \mathcal{J} (the original Jordan algebra structure).

For $g = (g_1, g_2) \in G$, we define an element of G_1 by

$$(2.19) \quad \mu_g = c(g_1) \det(g_2)^2 g_1 \in G_1.$$

Note that the map $g \rightarrow \mu_g$ is a homomorphism.

Theorem 2.20. If $x, y \in V_k^{\text{ss}}$, $g \in G_{k^{\text{sep}}}$ and $y = gx$, then $\mu_g : \mathcal{M}_{x k^{\text{sep}}} \rightarrow \mathcal{M}_{y k^{\text{sep}}}$ is an isomorphism of k -algebras

Proof. Let $X, Y \in \mathcal{J}_{k^{\text{sep}}} = \mathcal{M}_{x k^{\text{sep}}}$. Then

$$\begin{aligned} \mu_g(X) \circ_y \mu_g(Y) &= \Delta(gx)^{-1} S_{gx}(c(g_1) \det(g_2)^2 g_1(X), c(g_1) \det(g_2)^2 g_1(Y)) \\ &= \Delta(gx)^{-1} c(g_1)^2 \det(g_2)^4 S_{gx}(g_1(X), g_1(Y)) \\ &= \Delta(gx)^{-1} c(g_1)^2 \det(g_2)^4 c(g_1)^3 \det(g_2)^4 g_1(S_x(X, Y)) \text{ by Proposition 2.17} \\ &= \Delta(gx)^{-1} \Delta(x) c(g_1)^4 \det(g_2)^6 c(g_1) \det(g_2)^2 g_1(\Delta(x)^{-1} S_x(X, Y)) \\ &= c(g_1) \det(g_2)^2 g_1(X \circ_x Y) = \mu_g(X \circ_x Y). \end{aligned}$$

Therefore, μ_g is a homomorphism. Since μ_g is obviously bijective, it is an isomorphism. \square

The above theorem implies that if $g_x = (g_1, g_2) \in G_{k^{\text{sep}}}$ and $x = g_x w \in V_k^{\text{ss}}$, then $\mu_g^{-1}(\mathcal{M}_x) \subset \mathcal{J}_{k^{\text{sep}}} = \mathcal{M}_{w k^{\text{sep}}}$ is a k -form of \mathcal{J} . If $(\mathcal{M}, \mathfrak{n}) \in \text{JIC}(k)$ is the pair corresponding to x (see [3, Section 4.5]), \mathcal{M} was characterized in the same manner (see [3, (4.9)]). Therefore, Theorem 1.4 follows.

3. EQUIVARIANT MAP II

In this section, we prove Theorem 1.5.

Let

$$(3.1) \quad T_a(X, Y, Z) = 27D(a, a, X)D(a, a, Y)D(a, a, Z) - 24D(a, a, a)D(a \times X, a \times Y, a \times Z)$$

for $a, X, Y, Z \in \mathcal{J}$. Then the map

$$T : \mathcal{J} \ni a \mapsto T_a \in \text{Hom}_k(\mathcal{J} \otimes \mathcal{J} \otimes \mathcal{J}, k)$$

is k -linear.

Lemma 3.2. If $a \in \mathcal{J}$, $g \in G_1$ and $X, Y, Z \in \mathcal{J}$, then

$$T_{ga}(gX, gY, gZ) = c(g)^3 g(T_a(X, Y, Z)).$$

Proof. There exists $t \in \bar{k}$ such that $t^3 = c(g)$. We put $g_1 = t^{-1}g$. Then $g = tg_1$ and $g_1 \in H_1\bar{k}$. So,

$$\begin{aligned} D(ga \times gX, ga \times gY, ga \times gZ) &= t^6 D(g_1a \times g_1X, g_1a \times g_1Y, g_1a \times g_1Z) \\ &= t^6 D(\tilde{g}_1(a \times X), \tilde{g}_1(a \times Y), \tilde{g}_1(a \times Z)) \\ &= t^6 D(a \times X, a \times Y, a \times Z) \\ &= c(g)^2 D(a \times X, a \times Y, a \times Z). \end{aligned}$$

Since $D(ga, ga, gX) = c(g)D(a, a, X)$, etc., the lemma follows. \square

The following proposition plays a crucial role in proving Theorem 1.5.

Proposition 3.3. $T_e(X, Y, Z) = \text{Tr}((X \circ Y) \circ Z) (= \text{Tr}(X \circ (Y \circ Z)))$ for all $X, Y, Z \in \mathcal{J}$.

Proof. Since $D(e, e, e) = 1$,

$$T_e(X, Y, Z) = 27D(e, e, X)D(e, e, Y)D(e, e, Z) - 24D(e \times X, e \times Y, e \times Z).$$

Let

$$X = \begin{pmatrix} s_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & s_2 & x_1 \\ x_2 & \bar{x}_1 & s_3 \end{pmatrix}, Y = \begin{pmatrix} t_1 & y_3 & \bar{y}_2 \\ \bar{y}_3 & t_2 & y_1 \\ y_2 & \bar{y}_1 & t_3 \end{pmatrix}, Z = \begin{pmatrix} u_1 & z_3 & \bar{z}_2 \\ \bar{z}_3 & u_2 & z_1 \\ z_2 & \bar{z}_1 & u_3 \end{pmatrix} \in \mathcal{J}.$$

Note that

$$\begin{aligned} 6D(X, Y, Z) &= \sum_{\{i,j,k\}=\{1,2,3\}} s_i t_j u_k + \sum_{\{i,j,k\}=\{1,2,3\}} \text{tr}(x_i y_j z_k) \\ &\quad - \sum_i s_i \text{tr}(y_i \bar{z}_i) - \sum_i t_i \text{tr}(x_i \bar{z}_i) - \sum_i u_i \text{tr}(x_i \bar{y}_i). \end{aligned}$$

By (2.7) and (2.8),

$$e \times X = -\frac{1}{2} \begin{pmatrix} -(s_2 + s_3) & x_3 & \bar{x}_2 \\ \bar{x}_3 & -(s_1 + s_3) & x_1 \\ x_2 & \bar{x}_1 & -(s_1 + s_2) \end{pmatrix}$$

Therefore,

$$\begin{aligned} 48D(e \times X, e \times Y, e \times Z) &= [(s_2 + s_3)(t_1 + t_3)(u_1 + u_2) + (s_2 + s_3)(t_1 + t_2)(u_1 + u_3) \\ &\quad + (s_1 + s_3)(t_2 + t_3)(u_1 + u_2) + (s_1 + s_3)(t_1 + t_2)(u_2 + u_3) \\ &\quad + (s_1 + s_2)(t_2 + t_3)(u_1 + u_3) + (s_1 + s_2)(t_1 + t_3)(u_2 + u_3)] \\ &\quad - \sum_{\{i,j,k\}=\{1,2,3\}} \text{tr}(x_i y_j z_k) \\ &\quad - (s_2 + s_3)\text{tr}(y_1 \bar{z}_1) - (s_1 + s_3)\text{tr}(y_2 \bar{z}_2) - (s_1 + s_2)\text{tr}(y_3 \bar{z}_3) \\ &\quad - (t_2 + t_3)\text{tr}(x_1 \bar{z}_1) + (t_1 + t_3)\text{tr}(x_2 \bar{z}_2) + (t_1 + t_2)\text{tr}(x_3 \bar{z}_3) \\ &\quad - (u_2 + u_3)\text{tr}(x_1 \bar{y}_1) + (u_1 + u_3)\text{tr}(x_2 \bar{y}_2) + (u_1 + u_2)\text{tr}(x_3 \bar{y}_3) \\ &= 2 \sum_{\{i,j,k\}=\{1,2,3\}} s_i t_j u_k + 2(s_1 t_1 u_2 + \cdots) - \sum_{\{i,j,k\}=\{1,2,3\}} \text{tr}(x_i y_j z_k) \\ &\quad - \sum_{i \neq j} s_i \text{tr}(y_j \bar{z}_j) - \sum_{i \neq j} t_i \text{tr}(x_j \bar{z}_j) - \sum_{i \neq j} u_i \text{tr}(x_j \bar{y}_j). \end{aligned}$$

Since $6D(e, e, X) = 2(s_1 + s_2 + s_3)$, we have

$$\begin{aligned} 27D(e, e, X)D(e, e, Y)D(e, e, Z) &= (s_1 + s_2 + s_3)(t_1 + t_2 + t_3)(u_1 + u_2 + u_3) \\ &= \sum_i s_i t_i u_i + (s_1 t_1 u_2 + \cdots) + \sum_{\{i,j,k\}=\{1,2,3\}} s_i t_j u_k. \end{aligned}$$

Therefore,

$$(3.4) \quad \begin{aligned} T_e(X, Y, Z) = & \sum_i s_i t_i u_i + \frac{1}{2} \sum_{\{i,j,k\}=\{1,2,3\}} \text{tr}(x_i y_j z_k) \\ & + \frac{1}{2} \sum_{i \neq j} s_i \text{tr}(y_j \bar{z}_i) + \frac{1}{2} \sum_{i \neq j} t_i \text{tr}(x_j \bar{z}_i) + \frac{1}{2} \sum_{i \neq j} u_i \text{tr}(x_j \bar{y}_i). \end{aligned}$$

Replacing X, Y in (2.6) by $X \circ Y, Z$ respectively and taking the sum of diagonal entries, we can express $4\text{Tr}((X \circ Y) \circ Z)$ in the following manner:

$$\begin{aligned} & 2(2s_1 t_1 + \text{tr}(x_3 \bar{y}_3 + \bar{x}_2 y_2))u_1 \\ & + \text{tr}((s_1 y_3 + x_3 t_2 + \bar{x}_2 \bar{y}_1 + t_1 x_3 + y_3 s_2 + \bar{y}_2 \bar{x}_1) \bar{z}_3 + (s_1 \bar{y}_2 + x_3 y_1 + \bar{x}_2 t_3 + t_1 \bar{x}_2 + y_3 x_1 + \bar{y}_2 s_3) z_3) \\ & + 2(2s_2 t_2 + \text{tr}(x_1 \bar{y}_1 + \bar{x}_3 y_3))u_2 \\ & + \text{tr}((\bar{x}_3 \bar{y}_2 + s_2 y_1 + x_1 t_3 + \bar{y}_3 \bar{x}_2 + t_2 x_1 + y_1 s_3) \bar{z}_1 + (\bar{x}_3 t_1 + s_2 \bar{y}_3 + x_1 y_2 + \bar{y}_3 s_1 + t_2 \bar{x}_3 + y_1 x_2) z_3) \\ & + 2(2s_3 t_3 + \text{tr}(x_2 \bar{y}_2 + \bar{x}_1 y_1))u_3 \\ & + \text{tr}((x_2 t_1 + \bar{x}_1 \bar{y}_3 + s_3 y_2 + y_2 s_1 + \bar{y}_1 \bar{x}_3 + t_3 x_2) \bar{z}_2 + (x_2 y_3 + \bar{x}_1 t_2 + s_3 \bar{y}_1 + y_2 x_3 + \bar{y}_1 t_2 + s_3 \bar{x}_1) z_1). \end{aligned}$$

This coincides with 4 times (3.4). \square

The construction of the isotope defined for $a \in \mathcal{J}$ ($\det(a) \neq 0$) is given in [5, p.155]. Let

$$(3.5) \quad \begin{aligned} Q_a(X, Y) &= -6 \det(a) D(X, Y, a) + 9 D(X, a, a) D(Y, a, a), \\ \Phi_a(X, Y) &= 4 \det(a)^3 (X \times a) \times (Y \times a) + \frac{1}{2} (\det(a)^2 Q_a(X, Y) - Q_a(X, a) Q_a(Y, a)) a. \end{aligned}$$

Then the product structure and the associated bilinear form of the isotope corresponding to $a \in \mathcal{J}$ is given by

$$\begin{aligned} \mathcal{J}^2 \ni (X, Y) &\mapsto X \circ_a Y \stackrel{\text{def}}{=} \det(a)^{-4} \Phi_a(X, Y) \in \mathcal{J}, \\ \mathcal{J}^2 \ni (X, Y) &\mapsto \langle X, Y \rangle_a \stackrel{\text{def}}{=} \det(a)^{-2} Q_a(X, Y) \in k. \end{aligned}$$

(see [5, p.147, Proposition 5.6.2], [5, p.153, Proposition 5.8.2], [5, p.155, Proposition 5.9.2]).

Here we provide an alternative way to define the product structure on \mathcal{J}_a .

Definition 3.6. Suppose that $a \in \mathcal{J}, \det(a) \neq 0$. For $X, Y \in \mathcal{J}$, we define $X \circ_a Y \in \mathcal{J}$ to be the element such that $Q_a(X \circ_a Y, Z) = \det(a)^{-1} T_a(X, Y, Z)$ for all $Z \in \mathcal{J}$.

Since Q_a is a non-degenerate bilinear form, the definition of $X \circ_a Y$ is well-defined.

Theorem 3.7. If $a \in \mathcal{J}, \det(a) \neq 0, g_a \in GE_6 k^{\text{sep}}, a = g_a e \in \mathcal{J}$, then $\mathcal{J} \otimes k^{\text{sep}} \ni X \mapsto gX \in \mathcal{J}_a \otimes k^{\text{sep}}$ induces an isomorphism of k -algebras $\mathcal{J} \otimes k^{\text{sep}} \cong \mathcal{J}_a \otimes k^{\text{sep}}$.

Proof. Suppose that $X, Y, Z \in \mathcal{J} \otimes k^{\text{sep}}$. Since $\det(a) = c(g) \det(e) = c(g)$,

$$\begin{aligned} Q_a(g(X \circ Y), gZ) &= c(g)^2 Q_e(X \circ Y, Z) = c(g)^2 T_e(X, Y, Z) = c(g)^{-1} T_a(gX, gY, gZ) \\ &= \det(a)^{-1} T_a(gX, gY, gZ) = Q_a(gX \circ_a gY, gZ). \end{aligned}$$

Since this holds for all $Z \in \mathcal{J} \otimes k^{\text{sep}}, g(X \circ Y) = gX \circ_a gY$. Therefore, g induces an isomorphism of k -algebras $\mathcal{J} \otimes k^{\text{sep}} \cong \mathcal{J}_a \otimes k^{\text{sep}}$. \square

In the situation of Theorem 3.7, \mathcal{J}_a can be identified with the k -form $g^{-1}(\mathcal{J}_a) \subset \mathcal{J}_{k^{\text{sep}}}$. Therefore, this \mathcal{J}_a coincides with the isotope \mathcal{J}_a constructed in [5].

Let $m : V \rightarrow \mathcal{J}$ be the equivariant map in [3, Section 5]. Since m is defined by homogeneous polynomials of degree 4, $Q_{m(x)}, T_{m(x)}$ are defined by homogeneous polynomials of degrees 16, 24 respectively. Can we construct lower degree equivariant maps from V to the space of bilinear forms and trilinear forms on \mathcal{J} to give an alternative way to define the product structure on \mathcal{M}_x (see (2.18))? It does not seem so. For example, if there were such a quadratic form Q_x depending on x , the degree must be $16 - 12 = 4$. However, there seems to be only one equivariant map of degree

4 from $\wedge^2 \mathcal{J}$ to the space of bilinear forms on \mathcal{J} by the calculation of the software “LiE” ([1]) as follows.

```
> alt_tensor(2, [1,0,0,0,0,0])
1X[0,0,1,0,0,0]
> sym_tensor(2, [0,0,1,0,0,0])
1X[0,0,0,0,0,2] +1X[0,0,2,0,0,0] +1X[0,1,0,0,1,0] +1X[1,0,0,0,0,0] +
1X[1,1,0,0,0,0] +1X[2,0,0,0,0,1]
```

We can construct an equivariant map from $\wedge^2 \mathcal{J}$ to the space of bilinear forms on \mathcal{J} such that $w_1 \wedge w_2$ corresponds to the bilinear form

$$\mathcal{J}^2 \ni (X, X) \mapsto \sum_{i=1}^3 s_i^2 - \sum_{i=1}^3 \|x_i\| \in k$$

where as

$$\text{Tr}(X \circ X) = \sum_{i=1}^3 s_i^2 + \sum_{i=1}^3 \|x_i\|.$$

Therefore, we cannot obtain $\text{Tr}(X \circ X)$.

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